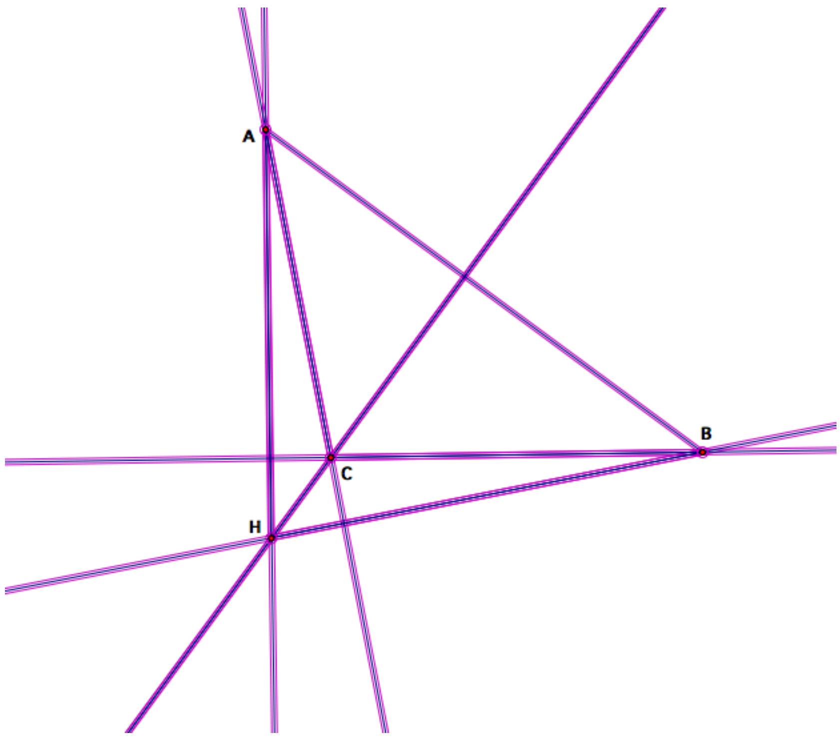
The University of Georgia
Department of Mathematics and Science Education
J. Wilson, EMAT 6680

EMAT 6680 - Assignment 8

By Brandon Samples

Question: I am going to explore orthocenters and prove a fact that I discovered.

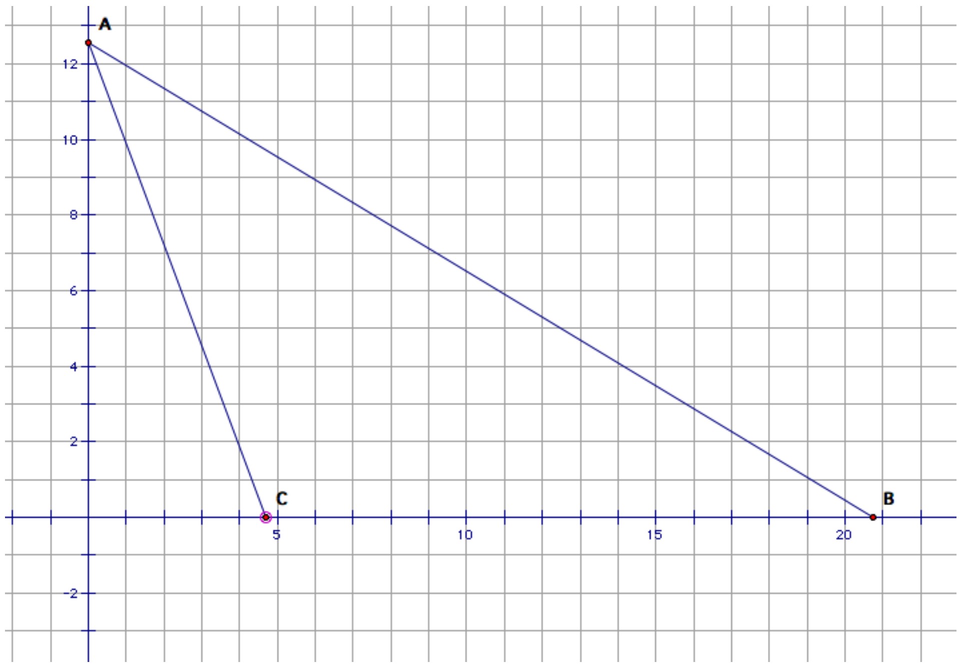
Let's begin by looking at the following figure:



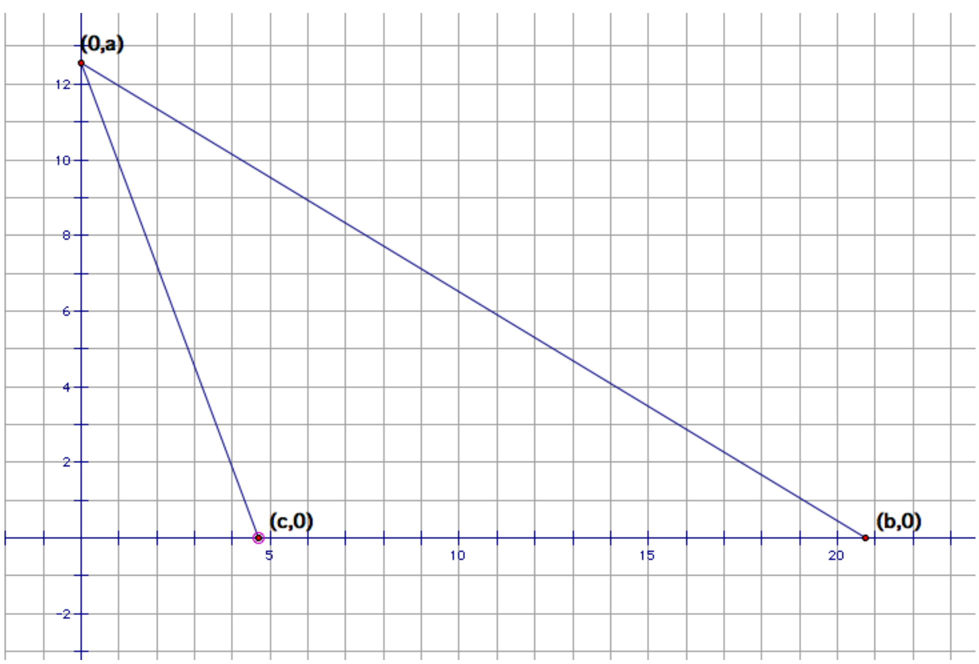
Given any triangle ABC, I constructed the orthocenter and labelled it H. Now, the interesting thing to think about is whether we will always know where the orthocenter of triangle HAB will be located. As we can see from the above figure, the orthocenter of triangle HAB is located at C, so we might conjecture that this happens always.

Theorem 1. *Given any triangle ABC with orthocenter H, the orthocenter of triangle HAB is located at C.*

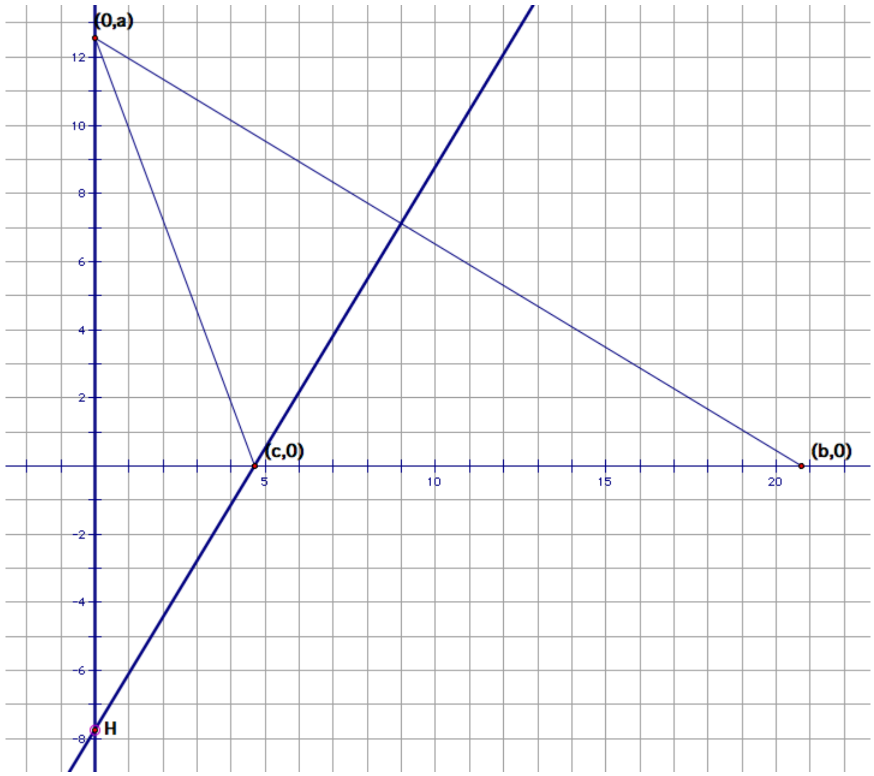
Proof. Take the triangle ABC and place the triangle on the coordinate axes such that the base of the triangle lies on the x-axis and the vertex opposite the base lies on the y-axis.



Next, we can label the vertices $(0, a)$, $(b, 0)$, and $(c, 0)$ based on their position in the xy-plane.



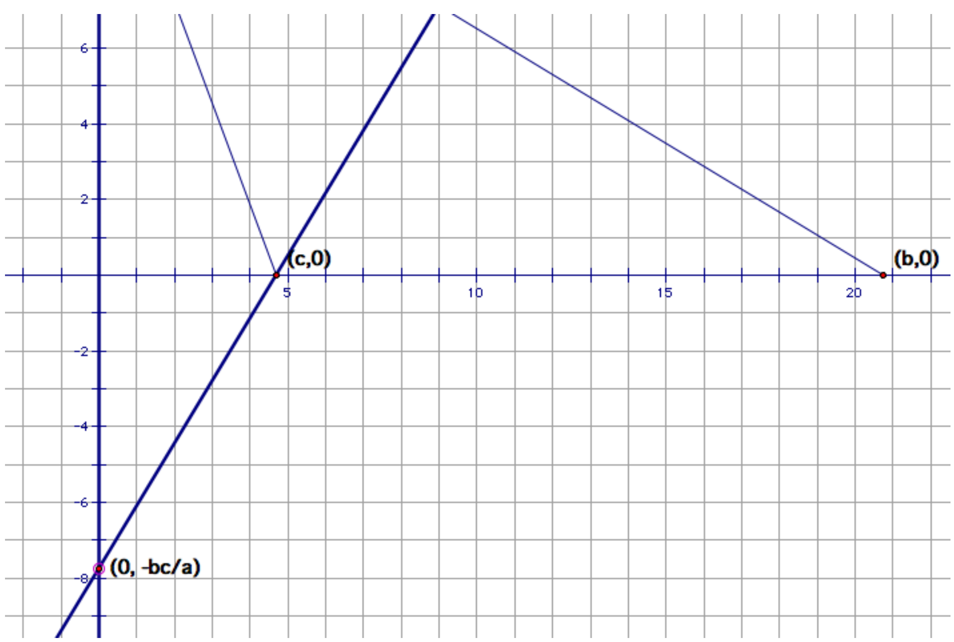
Construct the orthocenter to triangle ABC and label it H.



Now, let's figure out the coordinates of H. It's clear that the line through A perpendicular to the base is the y-axis, so the x-coordinate of H is $x = 0$. Next, the line through C perpendicular to the line through A and B is given by the equation

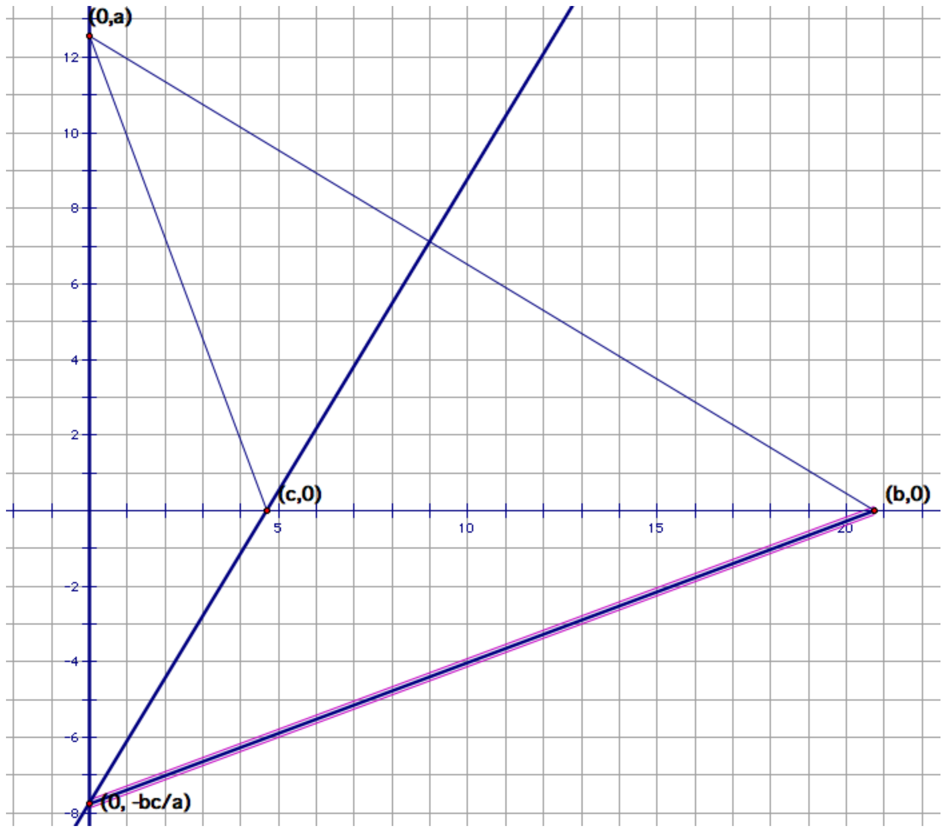
$$y = \frac{b}{a}(x - c).$$

Therefore, the y-coordinate of H is $y = \dfrac{-bc}{a}$.



Notice that the line perpendicular to the line through A and B passes through C, so let's show that the line through A perpendicular to the line through H and B also passes through C. Specifically, the line through H and B has slope $\frac{c}{a}$, so the line through A perpendicular to the line through H and B is given by the equation

$$y = \frac{-a}{c}(x - 0) + a = \frac{-a}{c}x + a.$$



Since the point $(c, 0)$ is a solution to the equation

$$y = \frac{-a}{c}x + a,$$

we see that C is the orthocenter of triangle HAB. □

Now, what's even more interesting to think about is that this proof is independent of our choice of A, B, and C, so what would we expect to happen if we had looked at the triangles HBC and HAC? If you think about the above proof, you can just think about rotating the triangle in the sense that you would choose a different side to play the role of the base. Then the above proof would apply in those cases, so B is the orthocenter of triangle HAC and A is the orthocenter of triangle HBC.